Moduli Spaces of Positive Curvature Metrics

DMV Jahrestagung - Sektion Geometrie & Topologie

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Curvature

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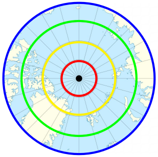
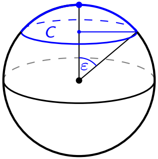
Notions of Curvature





Definition

$$\frac{l_P(\varepsilon)}{2\pi\varepsilon} = 1 - \frac{\mathrm{sec}_g(P)}{6}\varepsilon^2 + \mathcal{O}(\varepsilon^3) \qquad \text{Sectional Curvature}$$

$$\frac{\mathrm{d}\,\mathrm{vol}_g}{\mathrm{d}\,\mathrm{vol}_{\mathrm{eucl}}}(p + \varepsilon v) = 1 - \frac{\mathrm{Ric}(v, v)}{6}\varepsilon^2 + \mathcal{O}(\varepsilon^3) \qquad \text{Ricci Curvature}$$

$$\frac{\mathrm{vol}(B_\varepsilon(p) \subseteq M^d)}{\mathrm{vol}(B_\varepsilon(0) \subseteq \mathbb{R}^d)} = 1 - \frac{\mathrm{scal}_g(p)}{6(d+2)}\varepsilon^2 + \mathcal{O}(\varepsilon^3) \qquad \text{Scalar Curvature}$$

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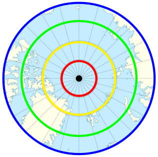

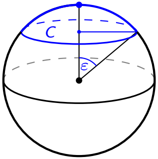

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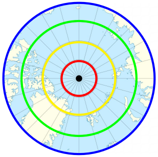

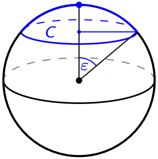

$C = 2\pi \sin(\varepsilon)$

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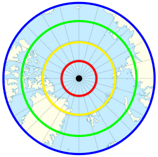

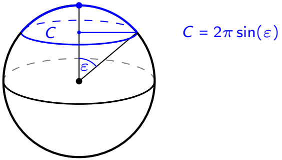

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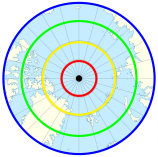

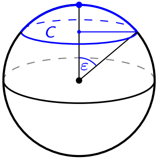

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Examples

Example (Positive Sectional Curvature Metrics)

- $S^n \subset \mathbb{R}^{n+1}$, $(\mathbb{C}P^n, g_{FS})$, $(\mathbb{H}P^n, g_{FS})$, $\mathbb{O}P^n$

Example (Positive Ricci Curvature)

- $S^n \times S^n$ or $\Sigma^n \in bP_{n+1}$ (**Wraith 2011**)

Example (Positive Scalar Curvature)

- $S^2 \times S^1$

Example (No positive curvature)

- T^n, $K3$-surfaces

Counting Geometries

Question

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Example (Hitchin)

Diffeomorphisms act via pull back

$$\mathrm{Diff}(M) \curvearrowright \mathrm{Riem}^{C}(M) \quad \text{via} \quad (\varphi, g) \mapsto \varphi^{*}g.$$

Hitchin '74: There is $[\varphi(t)] \in \pi_1(\mathrm{Diff}(S^8))$ such that

$$[\varphi(t)^{*}g_{\mathrm{round}}] \neq 0 \in \pi_1(\mathrm{Riem}^{\mathrm{scal}}(S^8))$$

Moduli Spaces

Observer Diffeomorphism

$$\mathrm{Diff}_{x_0}(M) := \{\varphi \in \mathrm{Diff}(M) \,:\, D_{x_0}\varphi = \mathrm{id}\} \curvearrowright \mathrm{Riem}^{C}(M) \text{ freely}$$

Versions of Moduli Spaces

$\mathfrak{M}^{C}(M) := \mathrm{Riem}^{C}(M)/\mathrm{Diff}(M)$	Moduli Space
$\mathfrak{M}^{C}_{x_0}(M) := \mathrm{Riem}^{C}(M)/\mathrm{Diff}_{x_0}(M)$	Observer M.S.
$\mathfrak{h}\mathfrak{M}^{C}(M) := \big(\mathrm{Riem}^{C}(M) \times E\mathrm{Diff}(M)\big)/\mathrm{Diff}(M)$	homotopy M.S.
$\mathfrak{h}\mathfrak{M}^{C}_{x_0}(M) := \big(\mathrm{Riem}^{C}(M) \times E\mathrm{Diff}_{x_0}(M)\big)/\mathrm{Diff}_{x_0}(M)$	h Obs. M.S.

Properties of Moduli Spaces

$$\begin{array}{ccc} \mathrm{Riem}^{C}(M)/\mathrm{Diff}_{x_0}(M) & \longrightarrow & \mathrm{Riem}^{C}(M)/\mathrm{Diff}(M) \\ \uparrow \simeq & & \uparrow \\ \mathrm{Riem}^{C}(M)//\mathrm{Diff}_{x_0}(M) & \longrightarrow & \mathrm{Riem}^{C}(M)//\mathrm{Diff}(M) \end{array} \tag{1}$$

$$\mathrm{Riem}^{C}(M) \longrightarrow \mathfrak{h}\mathfrak{M}(M) = \mathrm{Riem}^{C}(M)//\mathrm{Diff}(M) \longrightarrow B\mathrm{Diff}(M) \tag{2}$$

$$[X, \mathfrak{h}\mathfrak{M}(M)] \xrightarrow[\cong]{[f]\mapsto f^{*}g_{\mathrm{univ}}} \coprod_{[E]} \pi_0\big(\mathrm{Riem}^{C}_{\mathrm{vert}}(E)\big) \tag{3}$$

Main Theorem

Theorem (H.)

For all $n \geq 2$ and all odd $3 \leq j \leq n-1$, we have

$$\pi_2(\mathfrak{h}\mathfrak{M}^{\mathrm{sec}>0}_{x_0}(\mathbb{C}P^2), [g_{FS}]) \neq 0, \tag{1}$$

$$\pi_{2j}(\mathfrak{h}\mathfrak{M}^{\mathrm{sec}>0}_{x_0}(\mathbb{C}P^n), [g_{FS}]) \otimes \mathbb{Q} \neq 0. \tag{2}$$